



ANALYSIS AND SIMULATION OF BROADBAND AND CROSS-BAND PIM IN BASE STATION ANTENNAS

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1. Introduction

This white paper examines the basic theory of passive intermodulation (PIM) from a signal perspective, and also why real-world cases differ from the standard two-tone test used for passive devices such as filters and antennas. We'll also go into more detail about:

- What happens when you have more than two input signals?
- If you change input signals from single tone (CW) to modulated, what happens to PIM levels?
- What is the effect of the signal waveform on intermodulation products in bandwidth and spectrum power density?
- How can you estimate total intermodulation spectrum when you have a large number of input signals at different ports?

We'll also demonstrate our 3D PIM Calculator that lets you estimate PIM spectrum with simultaneous signals with different modulation at different input ports. The calculator can be used for analyzing multiband base station antennas, among other use cases and applications.

2. Basic PIM theory

Theory for PIM from a single source

There are multiple possible sources of PIM in a passive radio frequency (RF) product like a base station antenna or filter. These include:

- Metal-insulator-metal and metal-metal contact non-linearities (diode effect, tunneling)
- Thermionic emission, field break-through ("arc")
- Contact potential (weak diode generation between different metals)
- Ferromagnetic non-linearity
- Piezoelectric material non-linearity
- Thermo-electrical non-linearity (variation of conductivity with temperature)

You generally don't know which of these might be present in your antenna or filter. However, there are some common physical relationships that can help you work out what is causing PIM, such

as the Shockley Diode Equation¹, which describes the current-voltage relation in the circuit shown in Figure 1. This equation can be written as

(1)

$$y(t) = I_S (e^x - 1) \approx I_S (x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots)$$

using a Taylor series expansion of the exponential function around zero. This supports the commonly used model for PIM generation in the form of a polynomial representing the non-linear effect present:

(2)

$$y(t) = g_1 x + g_2 x^2 + g_3 x^3 + g_4 x^4 + g_5 x^5 \dots$$

The variables y and x could represent any real-valued circuit variable, like voltage or current.

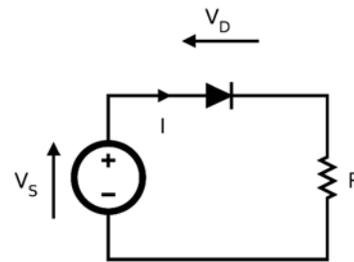


Figure 1: Example of a non-linear device, a diode modeled by a non-linear equation (1).

In most frequency division duplex (FDD) systems, the potential for PIM relates to two or more tones in a Tx band generating PIM in the associated Rx band. The industry standard test for PIM performance simplifies this to two *continuous wave* (CW) signals. One benefit of this is that the frequencies for PIM order $m + n$ are simple to calculate (see Appendix A):

$$f_{\{m+n\}} = m f_1 \pm n f_2$$

¹ See, e.g., https://en.wikipedia.org/wiki/Shockley_diode_equation.

$$I(t) = I_S \left(e^{\left(\frac{V_D}{nV_T}\right)} - 1 \right) \approx I_S \left(\left(\frac{V_D}{nV_T}\right) + \frac{1}{2} \left(\frac{V_D}{nV_T}\right)^2 + \frac{1}{6} \left(\frac{V_D}{nV_T}\right)^3 \dots \right),$$

I_S is the saturation current, V_D is the voltage across the diode, V_T is the thermal voltage, and n is the ideality factor. In our case, $y = I(t)$ and $x = V_D/(nV_T)$.

The most common case of PIM is the third order, where $|m| + |n| = 3$. However, it's worth remembering that, if you consider a fifth-order polynomial, you'd also get contributions at the frequencies of third-order products. For example:

$$2f_1 + f_1 - f_1 - f_2 = 2f_1 - f_2.$$

This is a complication when analyzing PIM signals—something we go into greater depth about in "Simulation of CW signals." We won't be covering these any further in this paper.

The equation above is suited to analyzing PIM on a single port with Tx signals in a single band. However, in real-world deployments, PIM can be generated by three tones, each in a different Tx band. The additional in-band PIM frequencies for the three-tone case are also straightforward; for PIM3, there are three relevant frequencies:

$$f_1 + f_2 - f_3, \quad f_1 + f_3 - f_2, \quad f_2 + f_3 - f_1$$

However, what is the amplitude of these PIM products relative to the two-tone case? Let's work out the coefficients for two-tone and three-tone cases and compare them.

Two-tone vs. three-tone PIM

Here we'll calculate the coefficients for the relevant PIM3 products for both cases, and compare their magnitudes. Using CW signals, we can expand the polynomial (2) and establish that the PIM3 signals have twice the magnitude for the three-tone case, corresponding to 6 dB in power (see Appendix A). However, if we set the total input power of the two and three tones equal, the difference is smaller, at 0.738 dB. Both cases are illustrated in Figure 2 below.

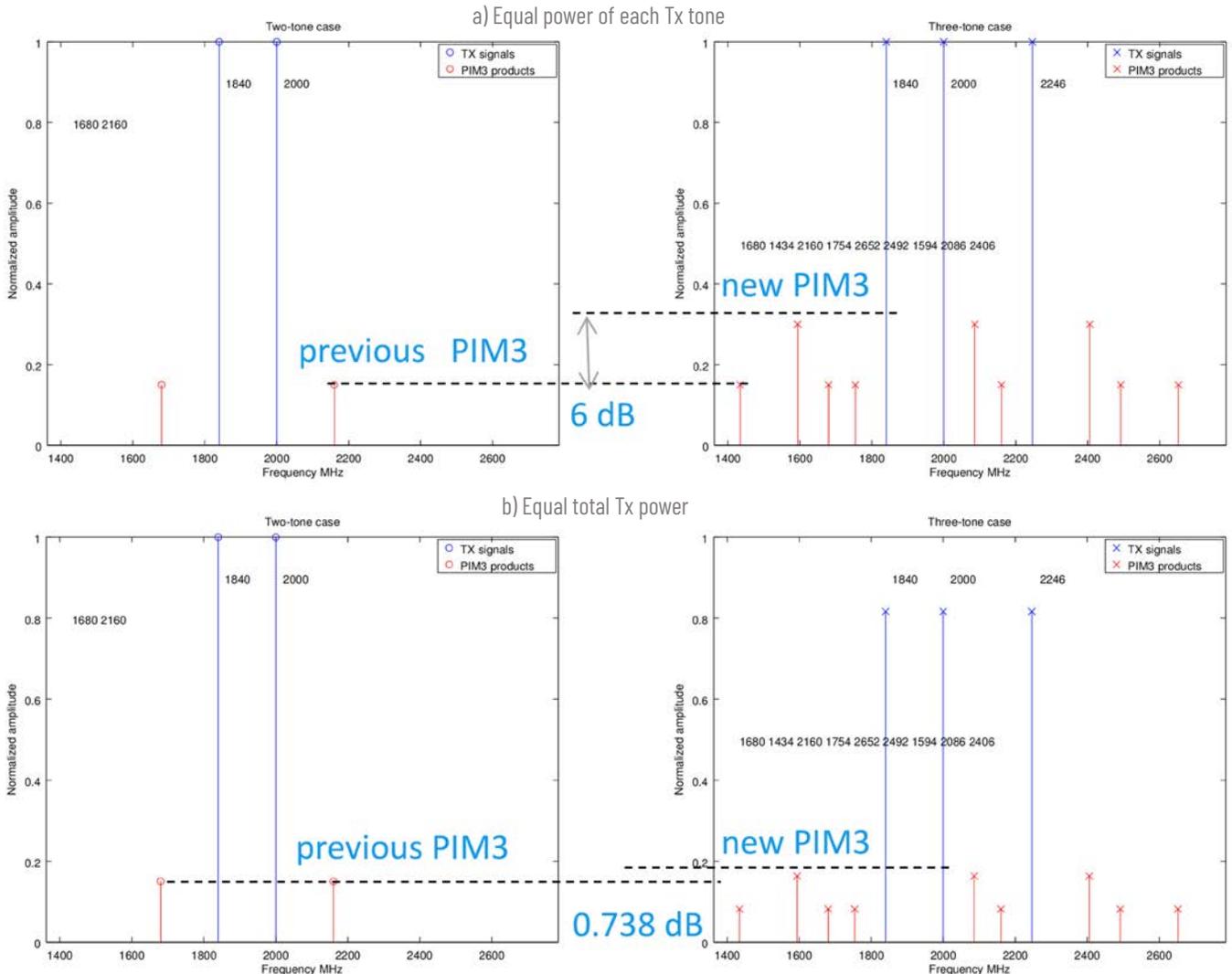


Figure 2: Comparison of two-tone and three-tone case

Total PIM power from two and three tones

As mentioned, the increase in PIM3 is small when you move from two- to three-tone. However, you must also factor in that the number of PIM3 products increases too. Appendix B shows that splitting the second CW tone into two CW tones with the same total power generates higher total PIM3 power when you include all in-band products. This is illustrated in Figure 3 where the PIM3 products are calculated from the simple polynomial:

$$y = g_1x + g_3x^3$$

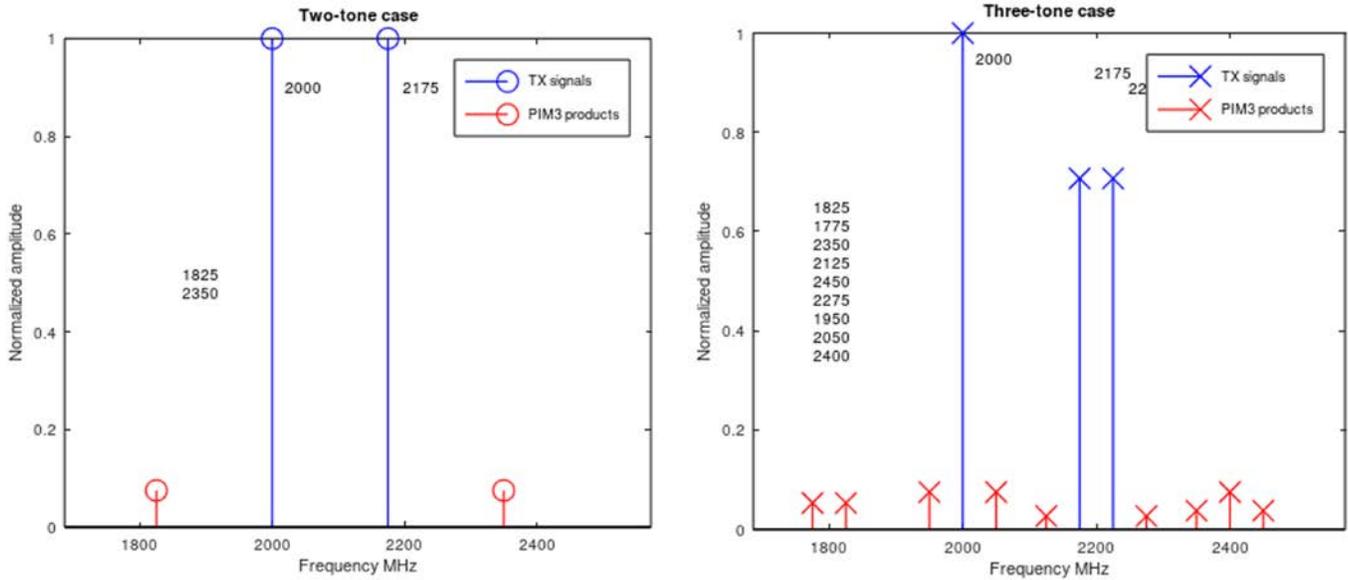


Figure 3: Illustration of two- and three-tone PIM3 products for the same input power.

with $g_1 = 1, g_3 = 0.1$. If we split the two Tx tones into three tones with the same power, the two in-band PIM3 products split up into a total of nine. Furthermore, close inspection of the PIM3 amplitude shows that the three strongest PIM3 products are the same amplitude as the two previous ones². The other 3 + 3 products have a relative amplitude of $1/\sqrt{2}$ or half of the two-tone case. Then, if we add the power from all the in-band PIM3 products, we have 3.76 dB higher power in the three-tone case³.

Broadband PIM

We've looked at PIM from CW tones with zero bandwidth, but what's the situation if we have broadband signals? With broadband, the bandwidth of the PIM products is larger than the Tx bandwidth, because multiplication in the time domain corresponds to convolution in the frequency domain. So, if you have Tx signals of equal bandwidth, a PIM3 product will be spread out over three times that bandwidth.

Changing from CW tones to a finite bandwidth typically leads to higher PIM power for a given input power and non-linear function. To understand this, we must consider the statistical distribution of the sum amplitude from a number of signals. If you only have two tones with amplitude a_0 , the total amplitude will always be $\leq 2 a_0$. But if you split these two tones into many smaller ones, as

in OFDM, the total amplitude approaches a Rayleigh distribution with no upper limit. This means the non-linear effect can produce more PIM power, similar to the previous section where three CW tones provided both stronger PIM3 products and higher total PIM3 power.

Next, let's consider signals with a finite non-zero spectrum. The spectrum of the intermodulation products can only be calculated analytically for a few cases, such as with a pulse in the spectral domain with constant phase. The change in spectrum for second- and third-order products is illustrated in Figure 4, and details of the calculation are included in Appendix C.

² We saw a similar result in the previous section "Two-tone vs. three-tone PIM" but, in that case, the three-tone combination resulted in 0.79 dB higher power.

³ This increase in power is independent of the actual PIM level, i.e., the value of the coefficients g_1 and g_3 .

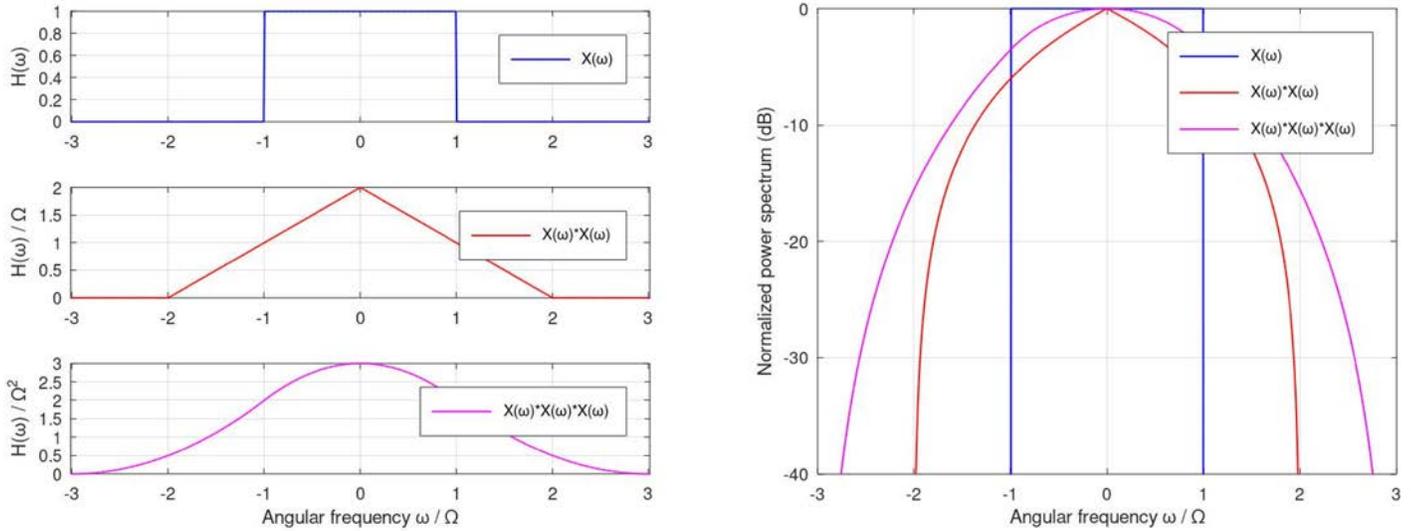


Figure 4: Illustrating the second- and third-order PIM spectrum from an input signal with pulse-shaped Fourier spectrum and bandwidth 2Ω .

Left: amplitude spectrum

Right: normalized power spectrum (dB)

Second order: $x^2(t) \rightarrow X(\omega) * X(\omega)$

Third order: $x^3(t) \rightarrow X(\omega) * X(\omega) * X(\omega)$

PIM level vs. input power

The polynomial expansion (2) of the non-linear transfer function suggests that the third-order PIM products will have an amplitude proportional to the cube x^3 of the input signal $x(t)$. That corresponds to a slope of 3 dB/dB in log-log scale. However, several published studies, plus our own results, show a smaller slope, i.e. the PIM power does not grow as fast with input power. In terms of the polynomial expansion, this corresponds to having alternating signs of the coefficients. As seen in Appendix A-B, the

PIM products resulting from the third- and fifth-order expansion partly coincide in frequency, which means the power slope for each PIM3 product depends on several coefficients. In fact, it actually does not stop there, because this is also true for the seventh- and ninth-order terms in the polynomial (2). Fitting the unknown polynomial coefficients to measured data can lead to a very good agreement, but a large number of polynomial terms is needed in general⁴.

⁴ See Fig. 4 in D. S. Kozlov, A. P. Shitvov, A. G. Schuchinsky and M. B. Steer, "Passive Intermodulation of Analog and Digital Signals on Transmission Lines With Distributed Nonlinearities: Modelling and Characterization," in **IEEE Transactions on Microwave Theory and Techniques**, vol. 64, no. 5, pp. 1383-1395, May 2016.

3. Cross-port PIM generation

To this point we've only talked about PIM generated from multiple signals present in the same RF path, i.e., on the same input port. In the real world, however, you will find that Tx signals exist simultaneously on multiple ports. This means:

- You have new frequency combinations and possible PIM products
- The overall voltage or current level increases throughout the antenna, and particularly on components common to multiple RF paths.

So, potential problems with cross-port PIM will be different depending on the source of the PIM. Figure 5 outlines two extreme examples: In both cases there is an antenna with two ports with an isolation = α^2 . For simplicity, we'll assume that this isolation is due to separation of the radiating elements, and that there is no insertion loss in the feed network from the input port to the radiator. As such, the mutual coupling between the ports is simply $|S_{21}|^2 = |S_{34}|^2 = 1/\alpha^2$.

In the first case (a), we can assume that the PIM is being generated by a source between the input port P1 and the radiating element connected to that port. That might be the input connector, a phase shifter, or a feed port for that particular antenna. We first measure the PIM signal y_a with two CW tones x_1 and x_2 on P1. Then, if we have signal x_2 moved to port P2, the signal x_2 from P2 will be attenuated⁵ to $x'_2 = x_2/\alpha$. So, what happens to the PIM level?

As shown in the equations in Appendix A, that depends on which PIM product we're talking about. The worst-case multi-port PIM happens when the PIM product is linearly proportional to the amplitude of x_2 , in which case a third-order PIM term of the total signal y_1 is reduced by a factor of α and its power by α^2 , i.e., the port-to-port isolation in our scenario⁶.

In the second case (b), we take the angle that the PIM source is common to both ports P1 and P2, which might mean a common part of a dual polarized radiating element, a shielding wall between two arrays, or something more external like a mounting bracket. Again we first measure the PIM signal y_b with two CW tones x_3 and x_4 present on P3. Then when we move the second signal x_2 to P4, we should expect to get the *same* level of the PIM signals y_3 and y_4 as we had for y_b .

It's worth noting that the isolation would likely reduce the overall PIM level as well in case (b), since antennas that are far apart are unlikely to result in multi-port PIM generation on any port.

However, it won't affect the *relative* PIM level of signal y_b , y_3 , and y_4 so, if we were to measure a certain value for y_b in the same port case, then that's also what we should expect when the signals are separated on different ports. Obviously, this is a worst-case scenario, so please don't worry—the PIM sources in the majority of cases would be fairly isolated from all ports.

So, how do you establish what makes a reasonable relationship between PIM levels y_1/y_a and $y_{3,4}/y_b$? One way is to make a cross-port measurement as outlined in Figure 5, but there are two challenges here:

- Only a small percentage of antennas fail PIM. And even when they do, the PIM level is relatively low—typically around -110 dBm.
- If we assume the cross-band PIM is reduced by a factor at least equal to the isolation as mentioned earlier, we end up around -140 dBm or so, a figure that is down in the noise floor for a typical PIM test system.
- To get more noticeable PIM values, you can add a PIM source such as a magnetic object or a ferrite material in the radiating aperture of the antenna. However, there's no way to know if this is representative for a real PIM source seen in a base station antenna.

So we are left with finding the rare antennas with significant PIM and then measuring them. We've done this for a number of antennas and have concluded that the typical case is Figure 5a, where cross-port PIM is reduced by at least the value of the port-to-port isolation $1/\alpha^2$, i.e.

$$|y_1|^2, |y_2|^2 < |y_a|^2/\alpha^2$$

A conservative assumption is to let

$$PIM_{xport} = PIM_{reflected} - Isolation \quad (dB)$$

So we will use this assumption in the numerical examples in the section "3D PIM Calculator GUI features and output format."

5 Ignoring any time delay since it does not affect this discussion.

6 Referring to App. A, equation (4) has in-band PIM terms proportional to $a_1^2 a_2$ as well as $a_2^2 a_1$ meaning that some PIM terms in our scenario will have an amplitude proportional to a_2/α and some to a_2^2/α^2 . The PIM level is thus reduced by α^2 or $(\alpha^2)^2$, which corresponds to 1x isolation or 2x isolation measured in dB.

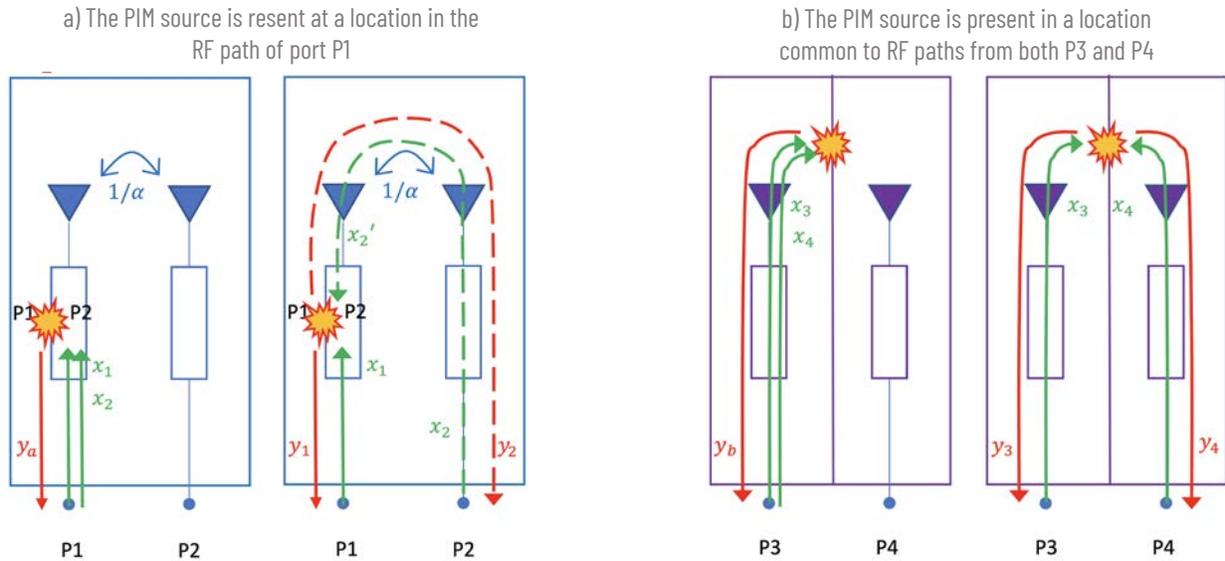


Figure 5: Two cases of cross-port PIM

4. 3-D broadband PIM calculator

Numerical simulation of PIM

So we've looked at PIM3 products from two or three CW signals, where we could calculate the PIM products analytically using an assumed model for PIM generation (i.e., the non-linear function represented by a polynomial expansion). In the real world, however, in a given use case there are more signals present, and they will also vary based on the finite bandwidth and in the modulation applied. Andrew's 3D PIM Calculator uses a hybrid time domain and frequency domain technique and combines the flexibility and accuracy of time domain simulation of various waveforms with the speed of a frequency domain simulation.

Our PIM simulation is based on these qualifiers:

- PIM generation model is a causal and memoryless process as in equation (1); the PIM at a given moment is a function of the input signal at that specific time⁷.
- PIM generation is constant over frequency⁸.
- When PIM products of different order overlap in frequency, we assume they add coherently in phase.

- If there are multiple TDD bands, it's assumed they are not synchronized. This means one TDD system can cause interference to another TDD system, along with the interference from FDD Tx signals.
- It's assumed that the modulated signal's peak-average ratio (PAR) is 8 dB.
- Modulated signal distortion level is linked to two-tone test result and specified by two-tone test result.

The graphical user interface (GUI) lets users set specific parameters:

- Antenna
 - a. Ports used: You can simulate single-band/port or multi-band/port
 - b. PIM distortion level⁹, both single-port and cross-port¹⁰
- Carriers configuration
 - a. Tx frequencies
 - b. Rx frequencies
 - c. Tx power levels
 - d. Bandwidths
 - e. Signal waveforms (CW, LTE, 5G)

⁷ An example of a real-world exception to this could be if the DUT sees thermal heating during the test, which may change the mechanical stress in the device and possibly the PIM level.

⁸ This is not true if there are multiple PIM sources adding in and out of phase at different frequencies, meaning a DUT has to pass the PIM test at the worst possible frequency. Our assumption can be seen as a worst case.

⁹ In-band refers to the reflected PIM3 level seen in a standard two-tone PIM test on a single port; cross-band is the PIM3 level generated when two signals are injected into different ports.

¹⁰ When Tx signals and distortion signal distributed on more than one port, this combination will be treated as cross-port.

The output from the 3D PIM Calculator includes data that's common to existing two-tone CW calculators, as well as other new features:

1. The number of PIM hits, and which Tx signals are contributing to each hit. Hits are listed based on strength so they can indicate direction of optimization.
2. The estimated desense of each Rx band. The receiver's sensitivity level is estimated by thermal noise plus an assumed 3 dB noise factor (NF) of the receivers. If the desense is less than 0.1 dB, then the corresponding hit will be ignored.
3. A plot of the receiver spectrum showing the PIM products up to fifth order, including respective bandwidth. The desense effect is calculated for each channel and we are able to calculate each sub-carrier's desense down to 30 kHz sub-carriers.

Simulation of CW signals

Here we'll look at examples of two continuous wave (CW) signals, similar to Figure 2 and Figure 3 in the previous chapter. Figure

6 and Figure 7 show the second- to fifth-order PIM products, including harmonics for two Tx signals with 20 W (+43 dBm) power centered at $f_1 = 700$ MHz and $f_2 = 960$ MHz. The PIM distortion level of each group is set to -110 dBm 2Tx @ +43 dBm. To show how the different PIM orders fall, we've plotted second and fourth order in Figure 6 and third and fifth in Figure 7 for different distortion orders and groups. Some points of interest from Figure 7:

- We can see the overlap between third- and fifth-order PIM products at common frequencies.
- The third-order harmonic products at $2T1 + T2$ and $2T2 + T1$ are $3^2 = 9 \Leftrightarrow 9.54$ dB stronger than the $3T1$ and $3T2$ products. This matches the theoretical result in Appendix A.
- For the fifth-order PIM products, the power differs by 6, 8, 14, and 20 dB respectively, which corresponds to the theoretical result in Appendix B.

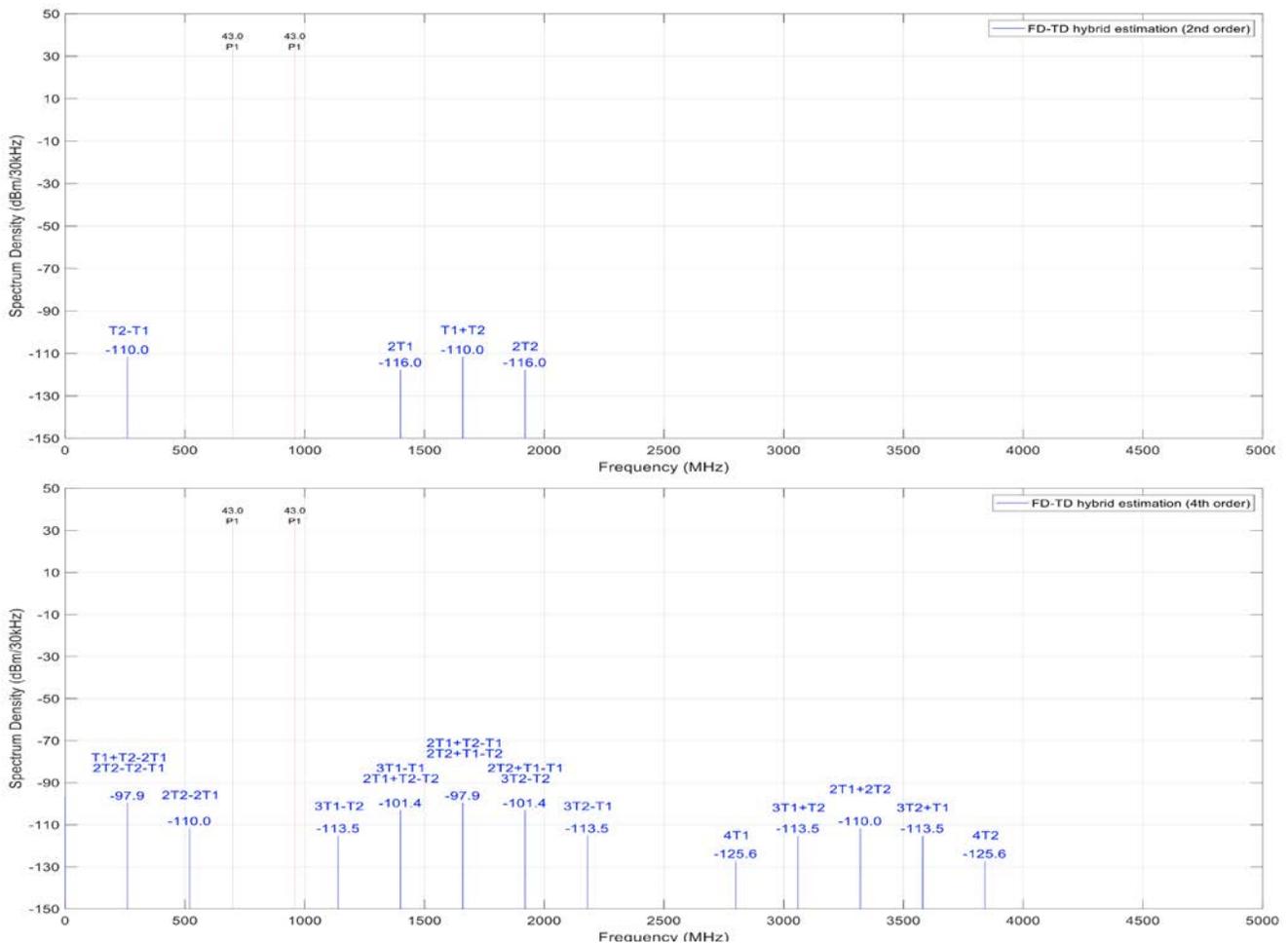


Figure 6: 3D PIM calculator example with 2x20 W CW signals, -110 dBm distortion level. The plots show second and fourth order.

Note the overlap at 260, 1400, 1660, and 1920 MHz.

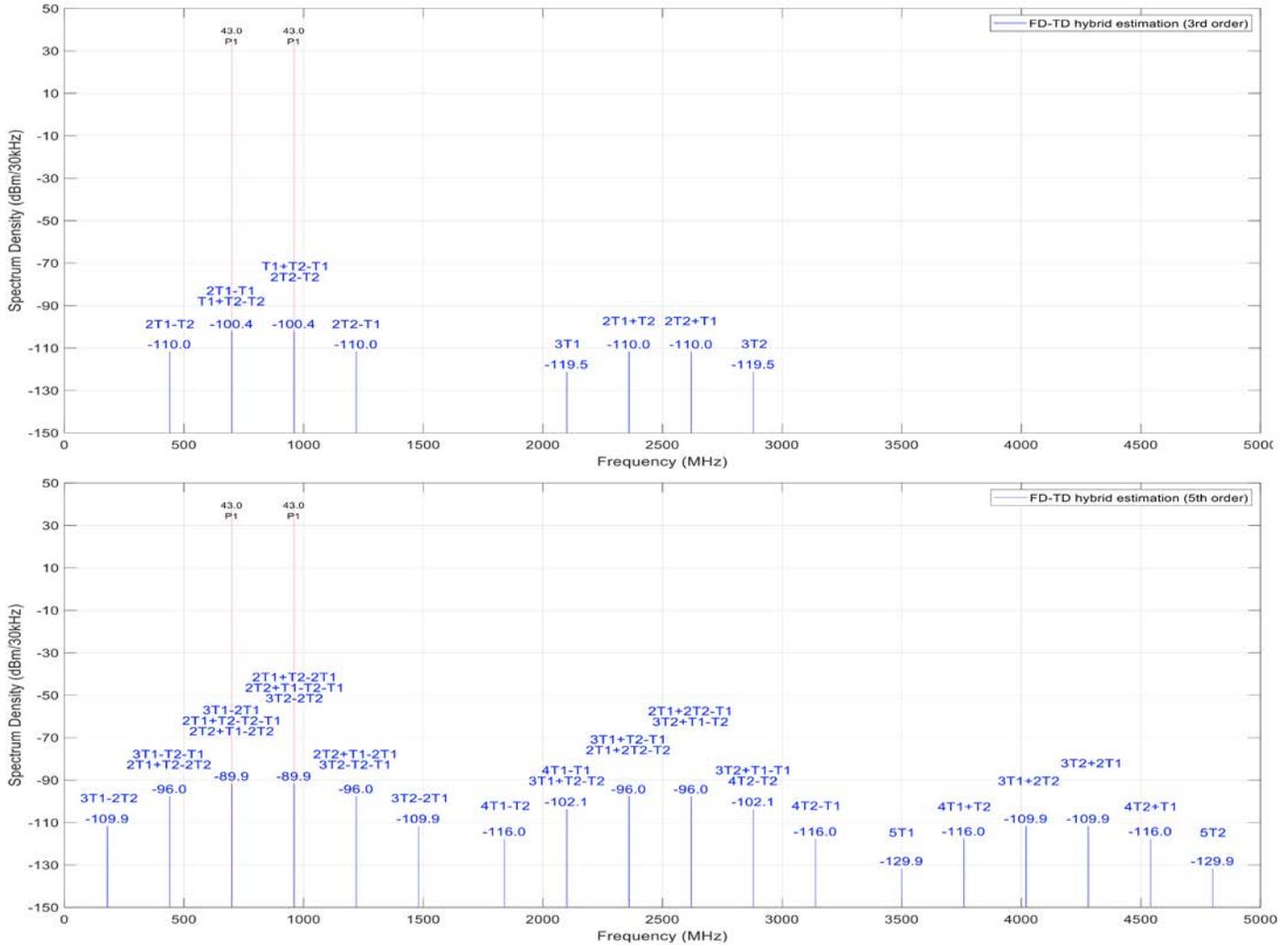


Figure 7: 3D PIM Calculator example with 2x20 W CW signals, -110 dBm distortion level. The plots show third and fifth order.
 Note the overlap at the overlapping frequencies, e.g., 440, 700, 960, 1220, 2100, 2360, 2620 and 2880 MHz.

Simulation of modulated signals

Those last examples could be calculated analytically, much as the examples in the “Two-tone vs. three-tone PIM” section, but that’s not the case if you switch modulated signals. In this example, we’ll keep the PIM3 distortion level at -110 dBm @ 2x43 dBm. Figure 8 and Figure 9 show the second- through fifth-order PIM products for 20 MHz and 5 MHz LTE signals centered at the same $f_1 = 700$ MHz and $f_2 = 960$ MHz, as previously. It shows that several PIM products overlap, something that is made clearer still in Figure 10.

In this example, the lower third-order PIM product falls at

$$2f_1 - f_2 = 440 \text{ MHz}$$

There is only one third-order combination at this frequency. But, if we consider fifth order, we get two new combinations adding to the total PIM level¹¹.

¹¹ The effect of fifth order obviously depends on the third- and fifth-order coefficients in equation (1), and these values need to be determined through measurement or other published results.

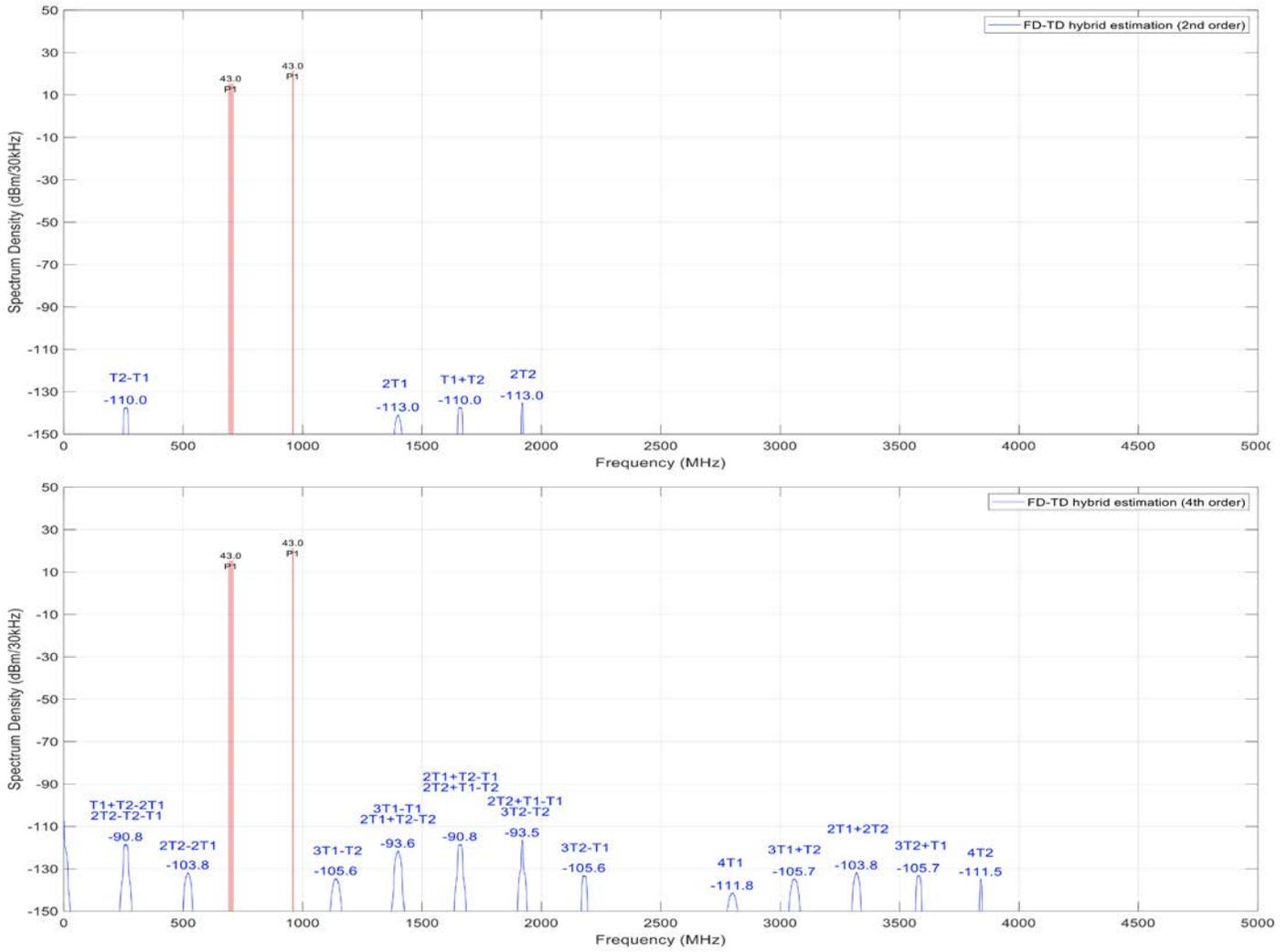


Figure 8: 3D PIM Calculator example showing two LTE signals, 20 MHz and 5 MHz bandwidth, PIM3 distortion level -110 dBm @ 2x43 dBm. The plots show the second- and fourth-order PIM.

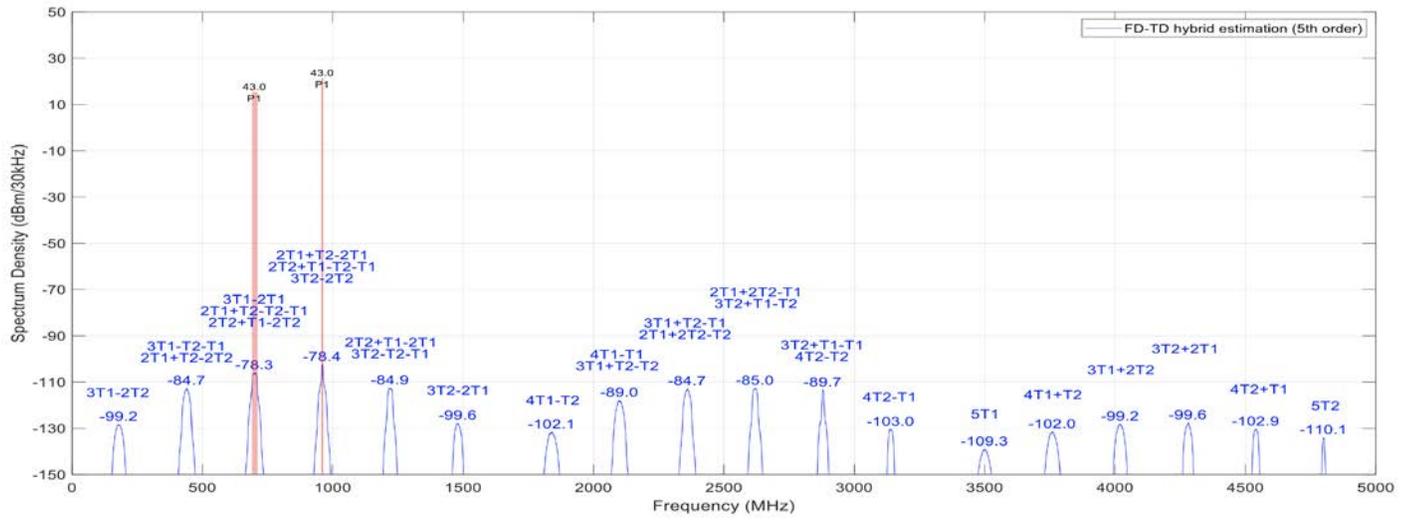
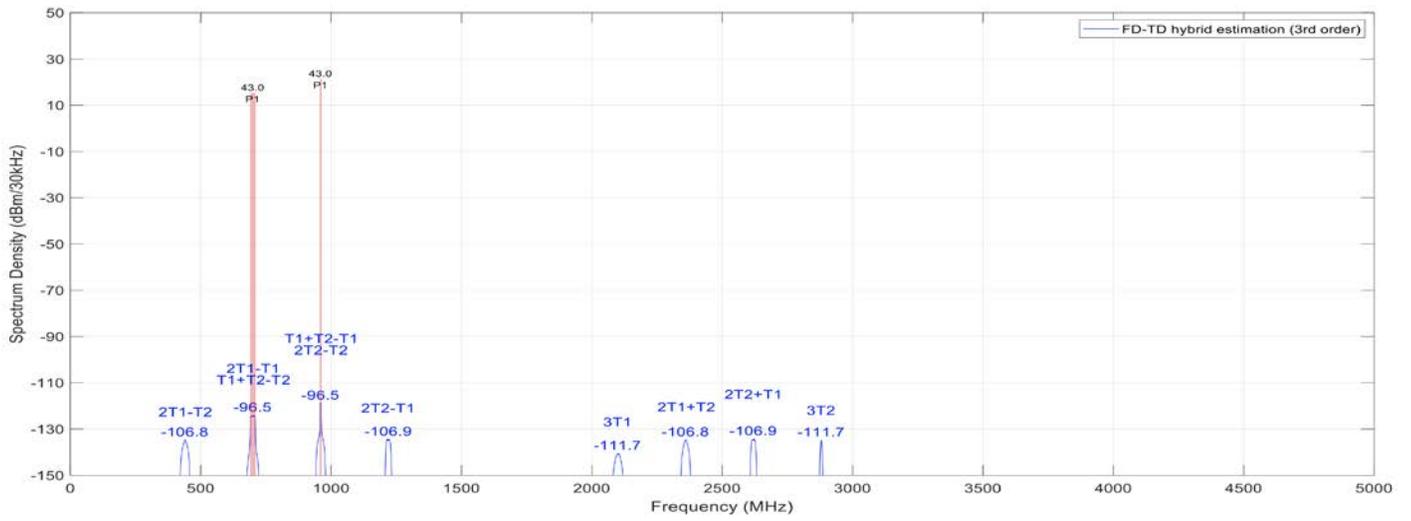


Figure 9: 3D PIM Calculator example showing two LTE signals, 20 MHz and 5 MHz bandwidth, PIM3 distortion level -110 dBm @ 2x43 dBm. The plots show the third- and fifth-order PIM.

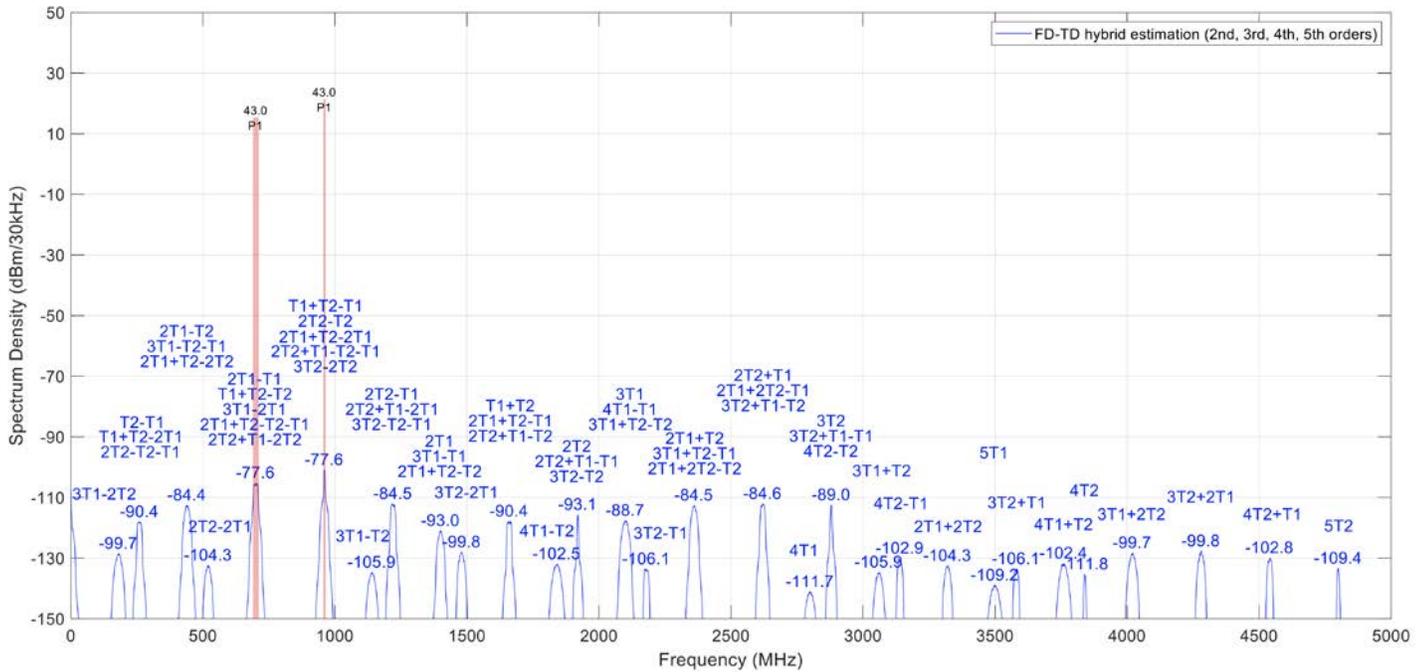


Figure 10: Plot showing all the PIM products from Figure 8 and Figure 9.
 Note: The overlap we get depends on the center frequency of the TX signals.

3D PIM Calculator GUI features and output format

Here we'll look at a real-world use case of an operator with resource spectrum in several bands, used on different antenna ports, and with different modulation of signals. It's worth pointing out that the calculator can be used for more than just base station antennas, but BSAs tend to be the best example for these

demonstrations as there are many signals. So first we set up the frequency bands and then choose a case similar to that in Figure 11. Here there are six different systems using a 2 x 4 ports configuration in the passive part of the antenna, plus a mMIMO system in the form of an active antenna system (AAS).

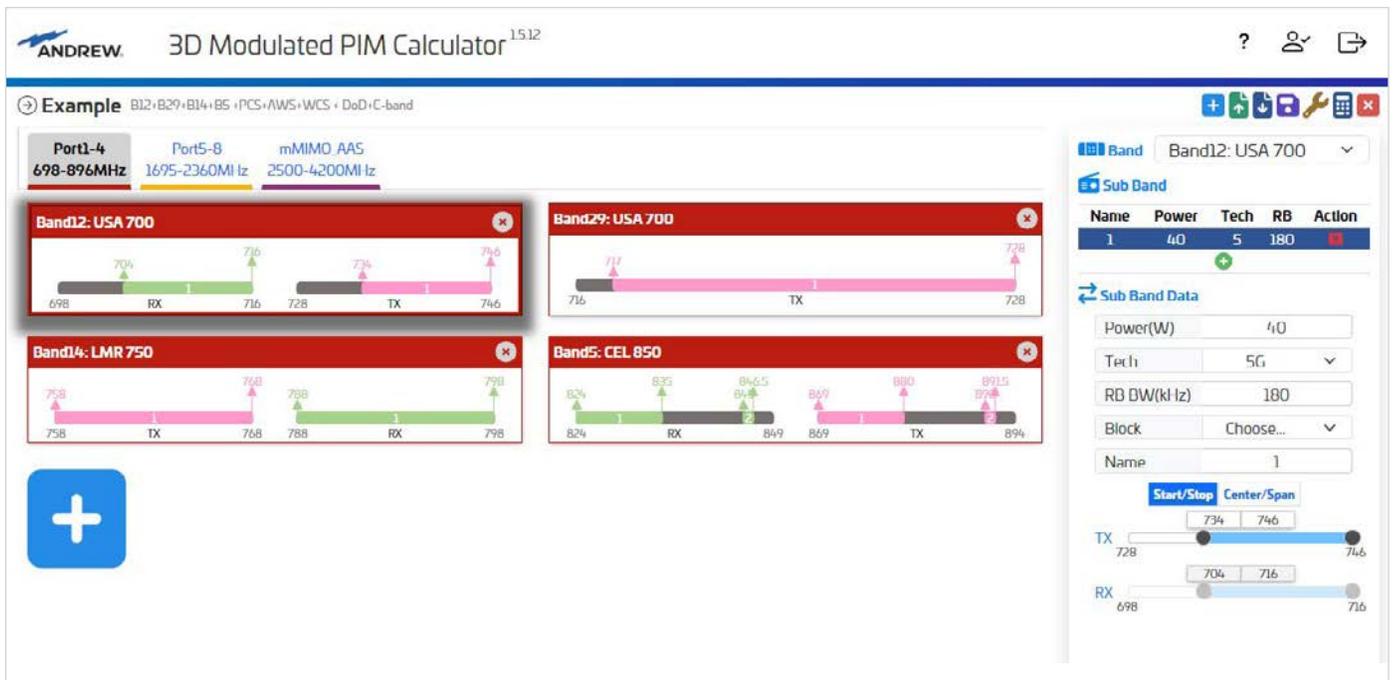


Figure 11: Frequency bands/power level definition

The PIM performance is pre-defined, using standards -110 dBm @ 2x43 dBm on CW signals for third-order PIM performance for a single port. For cross-port, the specification is -130 dBm,

assuming 20 dB isolation. For fourth order and fifth order, the PIM value used is -160 dBm—relatively low, but this is the lowest combination of many. The highest will be around 20 dB higher.



Figure 12: 3D PIM Calculator GUI after the simulation.

The results include the inference level and the desense of each receiver channel, e.g., 8.3 dB in Band 12-A block Rx.

Hit the Calculate button (🧮), wait for a few seconds, and the result will show the PIM spectrum as in Figure 12:

- The top plot shows the Tx bands as red bars, and Rx bands in blue. Power level is in dBm and band information, the number and center frequencies, are shown. The bars' width represents the bandwidth, and the frequencies are in order.
- The length of the red and blue bars indicates relative power levels.
- The right panel shows the worst port/band information by default, but users can select which band/port to check. Details include system noise, interference level and receiver desense, plus the contribution from each combination (only if single combination desense is higher than 0.1 dB).
- Bottom plot shows details in an enlarged chart. As well as the bars showing Tx and Rx bands, the solid red line represents total interference spectrum in dBm/30 kHz. Dashed lines represent PIM spectrum of each single combination contributing to PIM in the relevant Rx band.
- The receiver noise floor level is also shown for reference in dBm/30 kHz, calculated as the thermal noise floor plus a 3 dB receiver noise figure.
- When a specific combination is selected from the right panel list, the corresponding trace is highlighted so users can tell which is which.
- In this example, the worst case is B12-A with a desense of 8.3 dB caused by overlapping PIM products. The top contributor is from the Tx signal combination $2[B77-A]+[B29-A]-2[B77-A]$ ¹².

¹² This PIM product is in fact centered at the B29-A TX frequency, but the bandwidth of the B77 5G signal is so large that the PIM bandwidth extends into the B29 RX band. In this case, the choice of cross-band PIM distortion level in Figure 11 is crucial.

5. In conclusion

We've taken a look at PIM analysis for base station antennas and similar RF products where multiple signals are present in the same RF path, or in a common antenna aperture. And we found that, when you increase the number of Tx signals from two, as in the case of a standard PIM test, you'll see both higher peak PIM amplitude and higher total PIM power.

In the real world, if you want to predict where receiver performance may show a desense due to PIM products, you need a sophisticated tool that analyzes multiple signals on multiple antenna ports using modulated waveforms and transmit power across its occupied bandwidth. The Andrew 3D PIM Calculator is tailored to this job, and can be customized to operator-specific bands, technology, and RF power to predict system noise floor and corresponding desense within each receive band and taking into account actual performance of the passive path.

One particular concern with multi-port and multi-band systems is the risk of cross-port PIM. There are lots of parameters in play, so it isn't practical to determine them even for a single antenna configuration. In our opinion, a sensible assumption is that the cross-port PIM is equal to the in-band PIM minus the port-to-port isolation (in dB).

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Appendix

A. Third-order PIM for two and three tones

Begin from (2) using the first three terms:

$$y_{2T} = g_1 x + g_2 x^2 + g_3 x^3$$

and consider incident tones or *continuous wave (CW)* signals with arbitrary phase and amplitude defined by

$$\begin{aligned} x_p(t) &= a_p \cos(2\pi f_p t + \phi_p) = a_p \cos \theta_p \\ \theta_p &= 2\pi f_p t + \phi_p \end{aligned}$$

where f_p is the frequency and ϕ_p the phase of signal p . The phase is assumed to be constant in time for the interval in question. The total input signal is then

$$x(t) = x_1 + x_2 + \dots$$

Since we are interested primarily in the third-order products, we define third-order two-tone products:

(3)

$$y_{2T}''' = y_{2T} - g_1 x - g_2 x^2 = g_3 (a_1 \cos \theta_1 + a_2 \cos \theta_2)^3$$

We expand (3) using well-known algebra and trigonometry:

$$\begin{aligned} y_{2T}''' / g_3 &= (a_1 \cos \theta_1 + a_2 \cos \theta_2)^3 = \\ &= a_1^3 \cos^3 \theta_1 + 3a_1^2 a_2 \cos^2 \theta_1 \cos \theta_2 + 3a_1 a_2^2 \cos \theta_1 \cos^2 \theta_2 + a_2^3 \cos^3 \theta_2 = \left\{ \begin{array}{l} \cos^3 \theta = \frac{3 \cos \theta + \cos 3\theta}{4} \\ \cos^2 \theta = \frac{1 + \cos 2\theta}{2} \end{array} \right\} = \\ &= a_1^3 \frac{3 \cos \theta_1 + \cos 3\theta_1}{4} + a_2^3 \frac{3 \cos \theta_2 + \cos 3\theta_2}{4} \\ &\quad + 3a_1^2 a_2 \frac{1 + \cos 2\theta_1}{2} \cos \theta_2 + 3a_1 a_2^2 \cos \theta_1 \frac{1 + \cos 2\theta_2}{2} = \\ &= a_1^3 \frac{3 \cos \theta_1 + \cos 3\theta_1}{4} + a_2^3 \frac{3 \cos \theta_2 + \cos 3\theta_2}{4} + \frac{3}{2} a_1 a_2^2 \cos \theta_1 + \frac{3}{2} a_1^2 a_2 \cos \theta_2 \\ &\quad + \frac{3}{2} a_1^2 a_2 \cos 2\theta_1 \cos \theta_2 + \frac{3}{2} a_2^2 a_1 \cos \theta_1 \cos 2\theta_2 \end{aligned}$$

We expand the last two terms using the trigonometric identity¹³:

$$\begin{aligned} \frac{3}{2} a_1^2 a_2 \cos 2\theta_1 \cos \theta_2 + \frac{3}{2} a_2^2 a_1 \cos \theta_1 \cos 2\theta_2 &= \\ &= \frac{3}{4} a_1^2 a_2 [\cos(2\theta_1 - \theta_2) - \cos(2\theta_1 + \theta_2)] + \frac{3}{4} a_2^2 a_1 [\cos(2\theta_2 - \theta_1) + \cos(2\theta_2 + \theta_1)] \end{aligned}$$

The total list of third-order PIM products is then:

(4)

$$\begin{aligned} y_{2T}''' / g_3 &= (a_1 \cos \theta_1 + a_2 \cos \theta_2)^3 = \\ &= a_1^3 \left(\frac{3}{4} \cos \theta_1 + \frac{1}{4} \cos 3\theta_1 \right) + a_2^3 \left(\frac{3}{4} \cos \theta_2 + \frac{1}{4} \cos 3\theta_2 \right) + \frac{3}{2} a_1 a_2^2 \cos \theta_1 + \frac{3}{2} a_1^2 a_2 \cos \theta_2 \\ &\quad + \frac{3}{4} a_1^2 a_2 [\cos(2\theta_1 - \theta_2) - \cos(2\theta_1 + \theta_2)] + \frac{3}{4} a_2^2 a_1 [\cos(2\theta_2 - \theta_1) + \cos(2\theta_2 + \theta_1)] = \\ y_{2T}''' / g_3 &= \cos \theta_1 \left(\frac{3}{4} a_1^3 + \frac{3}{2} a_1 a_2^2 \right) + \cos 3\theta_1 \left(\frac{1}{4} a_1^3 \right) + \\ &\quad + [\cos(2\theta_1 - \theta_2) + \cos(2\theta_1 + \theta_2)] \left(\frac{3}{4} a_1^2 a_2 \right) + \\ &\quad + \cos \theta_2 \left(\frac{3}{4} a_2^3 + \frac{3}{2} a_2 a_1^2 \right) + \cos 3\theta_2 \left(\frac{1}{4} a_2^3 \right) + \\ &\quad + [\cos(2\theta_2 - \theta_1) + \cos(2\theta_2 + \theta_1)] \left(\frac{3}{4} a_2^2 a_1 \right) \end{aligned}$$

¹³ The identity is: $\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)]$

We can see from (4) and the definition of θ_p that the third-order frequencies follow the formula:

$$\omega_{\{m+n\}} = m \omega_1 \pm n \omega_2 \Leftrightarrow f_{\{m+n\}} = m f_1 \pm n f_2$$

Note that, even for the simple case $a_1 = a_2$, the amplitude of the different terms is proportional to factors of $3/2$, 1 , $3/4$, and $1/4$. In particular, the third-order harmonics $3\theta_1$ and $2\theta_1 + \theta_2$ differ by a factor of 3, corresponding to 9.56 dB.

Of particular interest are the two-tone in-band PIM¹⁴ products, which consist of the last two terms in (4):

$$(5) \quad \overline{y_{2T}'''} / g_3 = \frac{3}{4} a_1^2 a_2 [\cos(2\theta_1 - \theta_2)] + \frac{3}{4} a_2^2 a_1 [\cos(2\theta_2 - \theta_1)]$$

Now, moving on to the three-tone case, we define in the same way:

$$y_{3T}''' = y_{3T} - g_{1x} - g_2 x^2 = g_3 (b_1 \cos \theta_1 + b_2 \cos \theta_2 + b_3 \cos \theta_3)^3$$

We again only consider the third-order term:

$$\begin{aligned} \frac{y_{3T}'''}{g_3} &= (b_1 \cos \theta_1 + b_2 \cos \theta_2 + b_3 \cos \theta_3)^3 = \\ &= b_1^3 \cos^3 \theta_1 + b_2^3 \cos^3 \theta_2 + b_3^3 \cos^3 \theta_3 + \\ &+ 3b_1^2 b_2 \cos^2 \theta_1 \cos \theta_2 + 3b_1^2 b_3 \cos^2 \theta_1 \cos \theta_3 + \\ &+ 3b_1 b_2^2 \cos \theta_1 \cos^2 \theta_2 + 3b_1^2 b_3 \cos^2 \theta_2 \cos \theta_3 + \\ &+ 3b_1 b_3^2 \cos \theta_1 \cos^2 \theta_3 + 3b_2 b_3^2 \cos \theta_2 \cos^2 \theta_3 + \\ &+ 6b_1 b_2 b_3 \cos \theta_1 \cos \theta_2 \cos \theta_3 \end{aligned}$$

The **six blue terms** correspond to the third-order PIM from the two-tone case, while the **red term** results in a new PIM combination. Expanding these cosine terms, as for the two-tone case, we get the full list of PIM products and harmonics:

$$\begin{aligned} (6) \quad \frac{y_{3T}'''}{g_3} &= \dots = \frac{b_1^3}{4} (3 \cos \theta_1 + \cos 3\theta_1) + \frac{b_2^3}{4} (3 \cos \theta_2 + \cos 3\theta_2) + \frac{b_3^3}{4} (3 \cos \theta_3 + \cos 3\theta_3) + \\ &+ \frac{3b_1^2 b_2}{4} (\cos(2\theta_1 - \theta_2) + \cos(2\theta_1 + \theta_2) + 2 \cos \theta_2) + \frac{3b_1^2 b_3}{4} (\cos(2\theta_1 - \theta_3) + \cos(2\theta_1 + \theta_3) + 2 \cos \theta_3) \\ &+ \frac{3b_1 b_2^2}{4} (\cos(2\theta_2 - \theta_1) + \cos(2\theta_2 + \theta_1) + 2 \cos \theta_1) \\ &+ \frac{3b_2^2 b_3}{4} (\cos(2\theta_2 - \theta_3) + \cos(2\theta_2 + \theta_3) + 2 \cos \theta_3) \\ &+ \frac{3b_1 b_3^2}{4} (\cos(2\theta_3 - \theta_1) + \cos(2\theta_3 + \theta_1) + 2 \cos \theta_1) \\ &+ \frac{3b_2 b_3^2}{4} (\cos(2\theta_3 - \theta_2) + \cos(2\theta_3 + \theta_2) + 2 \cos \theta_3) \\ &+ \frac{3}{2} b_1 b_2 b_3 [\cos(\theta_1 + \theta_2 - \theta_3) + \cos(\theta_1 + \theta_3 - \theta_2) + \cos(\theta_2 + \theta_3 - \theta_1) + \cos(\theta_1 + \theta_2 + \theta_3)] \end{aligned}$$

14 We ignore the fundamental Tx tones $\cos \theta_1$, $\cos \theta_2$ as well as the third-order harmonics $\cos 3\theta_1$, $\cos 3\theta_2$. Note that the effect that third-order PIM increases also the fundamental tones also happens in the fifth-order PIM case. In that case we get an increase in the PIM level of the third-order products. See Appendix B.

As before, we are primarily concerned with in-band PIM3 products, so we keep only these terms:

$$(7) \quad \frac{\overline{y_{3T}''''}}{g_3} = \frac{3b_1^2b_2}{4}(\cos(2\theta_1 - \theta_2)) + \frac{3b_1^2b_3}{4}(\cos(2\theta_1 - \theta_3)) + \frac{3b_1b_2^2}{4}(\cos(2\theta_2 - \theta_1)) + \frac{3b_2^2b_3}{4}(\cos(2\theta_2 - \theta_3)) \\ + \frac{3b_1b_3^2}{4}(\cos(2\theta_3 - \theta_1)) + \frac{3b_2b_3^2}{4}(\cos(2\theta_3 - \theta_2)) \\ + \frac{3}{2}b_1b_2b_3[\cos(\theta_1 + \theta_2 - \theta_3) + \cos(\theta_1 + \theta_3 - \theta_2) + \cos(\theta_2 + \theta_3 - \theta_1)]$$

The green terms are similar to the two-tone case, having the same amplitude but more combinations, and the three red terms are the new PIM products for the three-tone case.

B. Fifth-order PIM products from two tones

For the two-tone case the fifth-order power expansion is:

$$y(t) = g_1x + g_2x^2 + g_3x^3 + g_4x^4 + g_5x^5,$$

We are interested in the fifth-order term:

$$(8) \quad y_{2T}^v = g_5x^5 = g_5(a_1 \cos \theta_1 + a_2 \cos \theta_2)^5$$

We expand (8) using well-known algebra and trigonometry:

$$\begin{aligned} y_{2T}^v/g_5 &= (a_1 \cos \theta_1 + a_2 \cos \theta_2)^5 = \\ &= a_1^5 \cos^5 \theta_1 + 5a_1^4 a_2 \cos^4 \theta_1 \cos \theta_2 + 10a_1^3 a_2^2 \cos^3 \theta_1 \cos^2 \theta_2 + 10a_1^2 a_2^3 \cos^2 \theta_1 \cos^3 \theta_2 \\ &+ 5a_1 a_2^4 \cos \theta_1 \cos^4 \theta_2 + a_2^5 \cos^5 \theta_2 \\ &= \frac{a_1^5}{16}(10 \cos \theta_1 + 5 \cos 3\theta_1 + \cos 5\theta_1) + \frac{5a_1^4 a_2}{8}(3 \cos \theta_2 + 4 \cos 2\theta_1 \cos \theta_2 + \cos 4\theta_1 \cos \theta_2) \\ &+ \frac{10a_1^3 a_2^2}{8}(3 \cos \theta_1 + \cos 3\theta_1)(1 + \cos 2\theta_2) + \frac{10a_1^2 a_2^3}{8}(1 + \cos 2\theta_1)(3 \cos \theta_2 + \cos 3\theta_2) \\ &+ \frac{5a_1 a_2^4}{8}(3 \cos \theta_1 + 4 \cos 2\theta_2 \cos \theta_1 + \cos 4\theta_2 \cos \theta_1) + \frac{a_2^5}{16}(10 \cos \theta_2 + 5 \cos 3\theta_2 + \cos 5\theta_2) \\ &= \frac{a_1^5}{16}(10 \cos \theta_1 + 5 \cos 3\theta_1 + \cos 5\theta_1) + \frac{5a_1^4 a_2}{8}(3 \cos \theta_2 + 4 \cos 2\theta_1 \cos \theta_2 + \cos 4\theta_1 \cos \theta_2) \\ &+ \frac{10a_1^3 a_2^2}{8}(3 \cos \theta_1 + 3 \cos \theta_1 \cos 2\theta_2 + \cos 3\theta_1 + \cos 3\theta_1 \cos 2\theta_2) \\ &+ \frac{10a_1^2 a_2^3}{8}(3 \cos \theta_2 + \cos 3\theta_2 + 3 \cos 2\theta_1 \cos \theta_2 + \cos 2\theta_1 \cos 3\theta_2) \\ &+ \frac{5a_1 a_2^4}{8}(3 \cos \theta_1 + 4 \cos 2\theta_2 \cos \theta_1 + \cos 4\theta_2 \cos \theta_1) + \frac{a_2^5}{16}(10 \cos \theta_2 + 5 \cos 3\theta_2 + \cos 5\theta_2) \end{aligned}$$

We expand the terms with products using $\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$:

$$\begin{aligned}
y_{2T}^v/g_5 &= \frac{a_1^5}{16}(10 \cos \theta_1 + 5 \cos 3\theta_1 + \cos 5\theta_1) + \frac{a_2^5}{16}(10 \cos \theta_2 + 5 \cos 3\theta_2 + \cos 5\theta_2) \\
&+ \frac{5a_1^4 a_2}{8} \left(3 \cos \theta_2 + 2(\cos(2\theta_1 - \theta_2) + \cos(2\theta_1 + \theta_2)) + \frac{1}{2}(\cos(4\theta_1 - \theta_2) + \cos(4\theta_1 + \theta_2)) \right) \\
&+ \frac{5a_2^4 a_1}{8} \left(3 \cos \theta_1 + 2(\cos(2\theta_2 - \theta_1) + \cos(2\theta_2 + \theta_1)) + \frac{1}{2}(\cos(4\theta_2 - \theta_1) + \cos(4\theta_2 + \theta_1)) \right) \\
&+ \frac{10a_1^3 a_2^2}{8} \left(3 \cos \theta_2 + \frac{3}{2}(\cos(\theta_1 - 2\theta_2) + \cos(\theta_1 + \theta_2)) \right) \\
&+ \cos 3\theta_1 + \frac{1}{2}(\cos(3\theta_1 - 2\theta_2) + \cos(3\theta_1 + 2\theta_2)) \\
&+ \frac{10a_1^2 a_2^3}{8} \left(3 \cos \theta_1 + \cos 3\theta_2 + \frac{3}{2}(\cos(2\theta_1 - \theta_2) + \cos(2\theta_2 + \theta_1)) + \frac{1}{2}(\cos(2\theta_1 - 3\theta_2) \right. \\
&\left. + \cos(2\theta_1 + 3\theta_2)) \right) + \frac{a_2^5}{16}(10 \cos \theta_2 + 5 \cos 3\theta_2 + \cos 5\theta_2)
\end{aligned}$$

Identifying terms and applying $\cos(-x) = \cos(x)$, we get:

$$\begin{aligned}
y_{2T}^v/g_5 &= \\
&\cos \theta_1 \left(\frac{5}{8} a_1^5 + \frac{15}{4} a_1^3 a_2^2 + \frac{15}{8} a_1 a_2^4 \right) + \cos 3\theta_1 \left(\frac{5}{16} a_1^5 + \frac{5}{4} a_1^3 a_2^2 \right) + \cos 5\theta_1 \left(\frac{1}{16} a_1^5 \right) + \\
&+ [\cos(2\theta_1 - \theta_2) + \cos(2\theta_1 + \theta_2)] \left(\frac{5}{4} a_1^4 a_2 + \frac{15}{4} a_1^2 a_2^3 \right) + \\
&+ [\cos(3\theta_1 - 2\theta_2) + \cos(3\theta_1 + 2\theta_2)] \left(\frac{5}{8} a_1^3 a_2^2 \right) + \\
&+ [\cos(4\theta_1 - \theta_2) + \cos(4\theta_1 + \theta_2)] \left(\frac{5}{16} a_1^4 a_2 \right) + \\
&+ \cos \theta_2 \left(\frac{5}{8} a_2^5 + \frac{15}{4} a_2^3 a_1^2 + \frac{15}{8} a_2 a_1^4 \right) + \cos 3\theta_2 \left(\frac{5}{16} a_2^5 + \frac{5}{4} a_2^3 a_1^2 \right) + \cos 5\theta_2 \left(\frac{1}{16} a_2^5 \right) + \\
&+ [\cos(2\theta_2 - \theta_1) + \cos(2\theta_2 + \theta_1)] \left(\frac{5}{4} a_2^4 a_1 + \frac{15}{4} a_2^2 a_1^3 \right) + \\
&+ [\cos(3\theta_2 - 2\theta_1) + \cos(3\theta_2 + 2\theta_1)] \left(\frac{5}{8} a_2^3 a_1^2 \right) + \\
&+ [\cos(4\theta_2 - \theta_1) + \cos(4\theta_2 + \theta_1)] \left(\frac{5}{16} a_2^4 a_1 \right)
\end{aligned}$$

It is worth noting that several of these PIM products coincide with the third-order products in Appendix C. In particular, this means the power of in-band products $\cos(2\theta_{1,2} \pm \theta_{2,1})$ will depend on the sign and magnitude of coefficients g_3 and g_5 .

If we consider the simple case with equal power of the input tones, $a_1 = a_2$, we can see that power of the fifth-order two-tone PIM products is proportional to:

Term	Amplitude / a_1^5	Power / $(a_1^5)^2$	Normalized power (dB)
$\cos \theta_{1,2}$	$\frac{5}{8} + \frac{15}{4} + \frac{15}{8}$	$\left(\frac{25}{4}\right)^2$	N/A*
$\cos 3\theta_{1,2}$	$\frac{5}{16} + \frac{5}{4}$	$\left(\frac{25}{16}\right)^2$	8.0
$\cos 5\theta_{1,2}$	$\frac{1}{16}$	$\left(\frac{1}{16}\right)^2$	-20
$\cos(2\theta_{1,2} \pm \theta_{2,1})$	$\frac{5}{4} + \frac{15}{8}$	$\left(\frac{25}{8}\right)^2$	14.0
$\cos(3\theta_{1,2} \pm 2\theta_{2,1})$	$\frac{5}{8}$	$\left(\frac{5}{8}\right)^2$	0
$\cos(4\theta_{1,2} \pm \theta_{2,1})$	$\frac{5}{16}$	$\left(\frac{5}{16}\right)^2$	-6.0

* Coincides with fundamental tones

C. Power of PIM products

Power of the individual PIM products

The two-tone PIM3 terms are of the form:

$$PIM3_{2T} = g_3^3 \frac{3}{4} a_1^2 a_2 \cos(2\theta_1 - \theta_2), \text{ etc.}$$

The three-tone case adds new terms combining three tones

$$PIM3_{3T} = g_3^3 \frac{3}{2} b_1 b_2 b_3 [\cos(\theta_1 + \theta_2 - \theta_3)], \text{ etc.}$$

If we have the same amplitude of the tones, $a_1 = a_2 = b_1 = b_2 = b_3$, the relative PIM3 amplitude is:

$$\frac{PIM3_{3T}}{PIM3_{2T}} = \frac{3/2}{3/4} = 2 \Leftrightarrow 6 \text{ dB}$$

If we instead normalize the power so $a_1^2 + a_2^2 = 2a^2 = b_1^2 + b_2^2 + b_3^2 = 3b^2$ so that $b = \sqrt{2/3} a$ we get the amplitude ratio:

$$\frac{PIM3_{3T}}{PIM3_{2T}} = \frac{3/2 \sqrt{2/3} \sqrt{2/3} \sqrt{2/3}}{3/4} = \frac{4 \cdot 2 \sqrt{2}}{2 \cdot 3 \sqrt{3}} = \frac{4\sqrt{2}}{3\sqrt{3}} = 1.089 \Leftrightarrow 0.738 \text{ dB}$$

Total power of all PIM products

For the two-tone case, using (5) and assuming the PIM3 frequencies do not coincide, the total power is the sum of the amplitude coefficients:

$$(8) \quad P_{2T} = \frac{9}{16} g_3^2 [a_1^4 a_2^2 + a_1^2 a_2^4] = \frac{9}{8} g_3^2 a_0^6$$

Likewise, under the same assumption for the three-tone case, we use (7) to calculate the total PIM power:

$$(9) \quad P_{3T} = g_3^2 \left[\frac{9}{16} (b_1^4 b_2^2 + b_1^2 b_3^2 + b_1^4 b_2^2 + b_2^4 b_3^2 + b_1^2 b_3^4 + b_2^2 b_3^4) + \frac{9}{4} 3 b_1^2 b_2^2 b_3^2 \right]$$

If we now let the power of the first tone equal one of the two tones above, and that the other two add up to the same power,

$$(10) \quad \begin{aligned} & b_1^2 = a_0^2 \\ & b_2^2 + b_3^2 = 2b_2^2 = a_0^2 \\ \text{we get} & P_{3T} = g_3^2 \left[\frac{9}{16} \left(\frac{a_0^4 a_0^2}{2} + \frac{a_0^4 a_0^2}{2} + \frac{a_0^2 a_0^4}{4} + \frac{a_0^4 a_0^2}{8} + \frac{a_0^2 a_0^4}{4} + \frac{a_0^2 a_0^4}{8} \right) + \frac{9}{4} 3 \frac{a_0^2 a_0^2 a_0^2}{4} \right] = \\ & = g_3^2 \left(\frac{9}{16} \cdot \frac{7}{4} + \frac{9}{4} \cdot \frac{3}{4} \right) a_0^6 = \frac{9}{8} \left(\frac{7}{8} + \frac{3}{2} \right) g_3^2 a_0^6 = \frac{19}{8} \cdot \frac{9}{8} g_3^2 a_0^6 \end{aligned}$$

This means the two cases' PIM3 power are related as follows:

$$P_{3T} = \frac{19}{8} \left(\frac{9}{8} g_3^2 a_0^6 \right) = \frac{19}{8} P_{2T} = 2.375 P_{2T}$$

which is equivalent to a 3.76 dB difference. This means that, for the same input power, splitting the power into three input signals results in more PIM power.

D. PIM spectrum from Fourier analysis (convolution)

Consider a signal $x(t)$ with finite bandwidth and let

$$X(\omega) \subset x(t)$$

be its Fourier transform. This means a polynomial of $x^n(t)$ corresponds to convolution of $X(\omega)$ with itself, e.g.,

$$x(t)x(t) \supset X(\omega) * X(\omega)$$

with the convolution defined by the integral

$$(4) \quad F(\omega) * G(\omega) = \int_{-\infty}^{\infty} F(\omega - u)G(u)du = \int_{\omega_1}^{\omega_2} F(\omega - u)G(u)du$$

where $\omega_{1,2}$ are the lower and upper limit of the support (non-zero region) of the two functions.

The above integral can't necessarily be worked out analytically, except in some special cases. One of these cases is when you have a signal with a constant spectrum over a given bandwidth 2Ω :

$$X(\omega) = \theta(\omega + \Omega) - \theta(\omega - \Omega) = \Pi_{2\Omega}(\omega)$$

where $\theta(\omega)$ is Heaviside's step function. This function $X(\omega)$ is shown in Figure 4 below and corresponds to a sinc function in the time domain, which doesn't necessarily provide the most practical example. However, it does demonstrate what happens when a broadband function is subject to a non-linear process, as defined by our polynomial (1). In this case we can calculate¹⁵ the Fourier spectrum of the square in the time domain $x(t)x(t)$, i.e., the convolution $X(\omega) * X(\omega)$, also shown in Figure 4. We can see that the bandwidth is now 4Ω . The amplitude has also increased in proportion to the bandwidth of $\Pi_{2\Omega}(\omega)$.

Next, we'll extend this process to calculating the third-order polynomial

$$x^3(t) = x(t)x(t)x(t) \supset X(\omega) * X(\omega) * X(\omega)$$

in a similar way, with the resulting spectrum shown in Figure 4. This time the bandwidth is tripled to 6Ω .

¹⁵ The details are omitted here, but it is straight-forward using the convolution integral (4) and variable substitution $\omega - u + \Omega = p$, etc. It can be shown that consecutive convolution of $X(\omega)$ with itself leads to a Gaussian distribution.

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